Some Late-time Asymptotics of General Scalar-Tensor Cosmologies

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Abstract

We study the asymptotic behaviour of isotropic and homogeneous universes in general scalar-tensor gravity theories containing a $p = -\rho$ vacuum fluid stress and other sub-dominant matter stresses. It is shown that in order for there to be approach to a de Sitter spacetime at large 4-volumes the coupling function, $\omega(\phi)$, which defines the scalar-tensor theory, must diverge faster than $|\phi_\infty - \phi|^{-1+\epsilon}$ for all $\epsilon > 0$ as $\phi \to \phi_\infty \neq 0$ for large values of the time. Thus, for a given theory, specified by $\omega(\phi)$, there must exist some $\phi_\infty \in (0, \infty)$ such that $\omega \to \infty$ and $\omega'/\omega^{2+\epsilon} \to 0$ as $\phi \to \phi_\infty$ in order for cosmological solutions of the theory to approach de Sitter expansion at late times. We also classify the possible asymptotic time variations of the gravitation 'constant' $G(t)$ at late times in scalar-tensor theories. We show that (unlike in general relativity) the problem of a profusion of "Boltzmann brains" at late cosmological times can be avoided in scalar-tensor theories, including Brans-Dicke theory, in which $\phi \to \infty$ and $\omega \sim o(\phi^{1/2})$ at asymptotically late times.

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1 Introduction

The cosmological approach to de Sitter space-time at large expansion times has assumed a double importance in the study of the universe. It provides a description of the evolution of the early universe after the prolonged gravitational influence of a slowly-evolving inflation field, whose energy density remains approximately constant for a significant interval of time. It also provides a good description of the late-time expansion dynamics of the universe in the presence of a dominant source of dark energy. We know that an exact de Sitter solution arises when the effective equation of state has the form $\rho + p = 0$, where p is the total isotropic pressure, and ρ is the total matter density. This can arise because of the presence of a single vacuum 'fluid' with this equation of state, as first suggested by Lemaître in 1933 [1], and employed in the construction of the steady state universe by Hoyle [2] and McCrea [3], and in the deduction of its stable asymptotic behaviour by Hoyle and Narlikar [4]. It could also result from the presence of an imperfect fluid which possess an effective equation of state

of this form, as is the situation with bulk viscosity [5], or from the presence of higher-order curvature terms in the gravitational Lagrangian beyond those sufficient to generate general relativity [6]. The correspondence between the situation in general relativity and such higher-order gravity theories can be understood in terms of the conformal equivalence of the two theories [7]. The situation becomes more complicated when other theories of gravity [8], which generalize Einstein's theory, are considered because the source of the field equations changes and a $\rho+p=0$ fluid stress no longer results in a simple de Sitter space-time. The most familiar example of this sort is the zero-curvature Brans-Dicke universe [9], where a $\rho+p=0$ fluid is no longer equivalent to the presence of an explicit cosmological constant term in the gravitational Lagrangian, and as such does not produce an asymptotic approach to the de Sitter metric [10, 11, 12]. Instead, power-law expansion of the isotropic expansion scale factor results, with the power becoming infinitely large as the theory approaches general relativity. This difference arises because, even when ρ is constant, the $G\rho$ term in the Friedmann equation falls in time, as t^{-2}, due to the fact that the gravitational 'constant' varies as $G \propto t^{-2}$ [13]. Higher-order gravity theories with Lagrangian contributions that are $O(1/R)$ also allow late-time acceleration to occur in a variety of different ways in the asymptotic limit of low 4-curvature, $R \to 0$.

In this paper, we will first consider the approach to the de Sitter metric in a generalized scalar-tensor theory [14], in order to gain some understanding of what type of accelerated expansion is possible in the situation where the simplest form of vacuum stress, with an equation of state $\rho+p=0$, dominates the expansion dynamics. In most scalar-tensor theories, the introduction of a vacuum stress will not result in a de Sitter spacetime. We shall be specifically interested in those theories which omit late times solutions in which the space-time approaches the de Sitter solution. We will classify such scalar-tensor theories of gravity by the speed of asymptotic approach to this de Sitter limit. In all such theories, G asymptotes towards a constant and non-zero value. In the second part of the paper, we consider the asymptotic evolution of scalar-tensor theories in which the gravitation 'constant' vanishes at late times. A general method for finding solutions of scalar-tensor cosmological models was given in by Barrow and Mimoso [15, 17, 18]. It is possible to adapt their generating-function method to study asymptotics, as was done for another problem by Deruelle et al [16], but we shall adopt a more direct approach. We classify the variation of $G(t)$ into three classes and then use our results to show how a solution of the 'Boltzmann brain' problem can be obtained in a wide range of scalar-tensor cosmologies, which includes the Brans-Dicke theory as a particular case.

In Section 2, we specify the general field equations for scalar-tensor theories defined by an arbitrary coupling function $\omega(\phi)$ of the scalar field. In Section 3, we classify the possible rates of approach to de Sitter spacetime for different behaviours of $\omega(\phi)$ and provide the conditions under which a generalized scalar tensor theory will omit a de Sitter limit. In Section 4 we consider the asymptotic forms for the possible evolutions of $G(t)$ in scalar-tensor cosmologies when a vacuum stress is present, using the characteristic $G \propto t^{-2}$ behaviour of Brans-Dicke theory as a benchmark. We show how theories in which $\phi \to \infty$ asymptotically, and the coupling function of the theory, $\omega(\phi)$, asymptotes to a sufficiently large value, or $\to \infty$ more slowly than $\phi^{1/2}$, will not suffer from the 'Boltzmann brain' problem that besets late-time cosmological evolution in general relativistic cosmologies. In Section 5, we summarize and discuss our results.

2 Field Equations

Consider a generalized scalar-tensor theory of gravity described by the action:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \omega(\phi)\frac{\partial_a\phi\partial^b\phi}{\phi}\right) + S_m(g_{ab}, \psi_m),$$

where we have set $c = \hbar = 1$; S_m is the matter action, ψ_m labels the matter fields. g_{ab} is the metric, ϕ is a scalar field and $\omega(\phi)$ is an arbitrary function that must be specified to fix the gravity theory. We define the energy-momentum tensor of matter to be T_{ab}. By varying the action with respect to the metric, and then with respect to ϕ, we obtain the field equations and conservation equations:

$$G_{ab} = \tfrac{8\pi}{\phi}T_{ab} + \tfrac{\omega}{\phi^2}\left(\partial_a\phi\partial_b\phi - \tfrac{1}{2}g_{ab}(\partial\phi)^2\right) + \tfrac{1}{\phi}\left(\nabla_a\nabla_b\phi - \Box\phi\right), \tag{1}$$

$$\Box\phi + \tfrac{\omega'}{3+2\omega}\left(\nabla\phi\right)^2 = \tfrac{8\pi T}{3+2\omega}, \tag{2}$$

$$T^a{}_{b;a} = 0. \tag{3}$$

The familiar Brans-Dicke theory arises when the arbitrary coupling function $\omega(\phi)$ is taken to be a constant. Under the assumptions of homogeneity and isotropy and spatial flatness the metric becomes:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right].$$

Here k determines the curvature of the $t = \text{const}$ slices. We assume that the universe is filled with some vacuum 'dark energy' with density ρ_0 and equation of state $\rho_0 = -p_0$, and that the rest of the matter has energy density $\rho_1 = \rho_0 K(y)$ and pressure p_1, where $y = \ln a$ and $\lim_{y\to\infty} K(y) = \lim_{y\to\infty} K_y(y) = 0$. The conservation equations require that $\rho_0 = \text{const}$ and that $p_1/\rho_0 = -\left(K'/3 + K\right)$. The gravitational field equations then reduce to:

$$\frac{3\dot{a}^2}{a^2} = \frac{8\pi\rho_0}{\phi}(1+K) + \frac{\omega(\phi)}{2}\frac{\dot{\phi}^2}{\phi^2} - \frac{3\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{3k}{a^2}, \tag{4}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{8\pi\rho_0}{\phi}(1 + K_y/3 + K) - \frac{\omega(\phi)}{2}\frac{\dot{\phi}^2}{\phi^2} - \frac{\ddot{\phi}}{\phi} - \frac{2\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{k}{a^2}. \tag{5}$$

3 The De Sitter Limit

Proposition 3.1. *In order for there to be an approach to de Sitter at large 4-volumes, where $\phi \to \phi_\infty$, the coupling function $\omega(\phi)$ must diverge faster than $|\phi_\infty - \phi|^{-1+\epsilon}$ for all $\epsilon > 0$ as $\phi \to \phi_\infty \neq 0$; $\phi_\infty < \infty$.*

We define $\dot{a}/a \equiv H = H_\infty/(1-F(y))$, $\phi_\infty = 8\pi\rho_0/3H_\infty^2$, $k = \gamma H_\infty^2$, and $\varphi = (\phi_\infty - \phi)/\phi_\infty$. The essential Einstein equations can then be rearranged to read:

$$\begin{aligned}\frac{2F_y}{1-F}\left[1 - \varphi - \frac{1}{2}\varphi_y\right] &= \varphi_{yy} - \varphi_y - \frac{\omega\varphi_y^2}{(1-\varphi)} \\ &\quad +(1-F)^2\left(K_y + 2\gamma(1-\varphi)e^{-2y}\right),\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\omega\varphi_y^2}{6(1-\varphi)} &= 2F - F^2 - \varphi - \varphi_y \\ &\quad -(1-F)^2\left(K - \gamma(1-\varphi)e^{-2y}\right),\end{aligned} \tag{7}$$

These equations are equivalent to the pair:

$$\begin{aligned}
\frac{2F_y}{1-F}\left[1-\varphi-\frac{1}{2}\varphi_y\right]+6(2F-F^2) &= \varphi_{yy}+5\varphi_y+6\varphi \\
&\quad +(1-F)^2\left(K_y+6K-4\gamma(1-\varphi)e^{-2y}\right) && (8)\\
-\frac{\omega_\varphi\varphi_y^2}{3+2\omega} &= \frac{(12+12K+3K_y)(1-F)^2}{3+2\omega} && (9)\\
&\quad +\varphi_{yy}+\varphi_y\left(3+\frac{F_y}{1-F}\right).
\end{aligned}$$

For there to be a de Sitter limit we need $\lim_{y\to\infty} H \to H_\infty = \text{const}$ which implies that $\lim_{y\to\infty} F = 0$ and also $\lim_{y\to\infty} F_y = 0$. It is clear from Eq. (8) that $F \to 0$, $F_y \to 0$ implies that $\varphi_{yy}+5\varphi_y+6\varphi \to 0$ because $\lim_{y\to\infty} K(y) = \lim_{y\to\infty} K_y(y) = 0$, and so φ_{yy}, φ_y , $\varphi \to 0$. In order to prove the proposition 3.1, we solve Eqs. (6-9) asymptotically in the limit $y \to \infty$ for general $F(y)$. We then find the asymptotic form that $\omega(\phi)$ must take for de Sitter spacetime to emerge in this limit.

We define:

$$M(y) = \frac{1}{(1-F)^2} - 1 - \left(K-\gamma e^{-2y}\right).$$

We then have:

$$\frac{2F_y}{1-F}+6(2F-F^2)-(1-F)^2\left(K_y+6K-4\gamma e^{-2y}\right) = (1-F)^2(N_y+6N),$$

and so Eqs. (6) and (8) become:

$$\begin{aligned}
(1-F)^2 M_y(y) &= \varphi_{yy}-\varphi_y\left[1-\frac{F_y}{1-F}\right]-\frac{\omega\varphi_y^2}{1-\varphi} && (10)\\
&\quad +\varphi\left[\frac{2F_y}{1-F}-2\gamma(1-F)^2e^{-2y}\right],\\
(1-F)^2\left(M_y+6M\right) &= \varphi_{yy}+\left(5+\frac{F_y}{1-F}\right)\varphi_y && (11)\\
&\quad +\left(6+4\gamma(1-F)^2e^{-2y}+\frac{2F_y}{1-F}\right)\varphi.
\end{aligned}$$

We consider three cases: $\lim_{y\to\infty} M_y/M = 0$, $\lim_{y\to\infty} M_y/M = -q$ for some $0 < q < \infty$ and $\lim_{y\to\infty} M_y/M \to 0$. If $K = \gamma = 0$ then these cases represent, respectively: slower than power-law approach (of $H - H_\infty$) to de Sitter, power-law approach to de Sitter, and faster than power-law approach to de Sitter.

3.1 Slow approach: $M_y/M \to 0$

We begin by considering those cases in which $M_y/M \to 0$ as $y \to \infty$. If $K = \gamma = 0$ this would imply $F_y/F \to 0$ and so $|H - H_\infty| \to 0$ more slowly than a^{-r} for any $r > 0$ i.e. $a^r\,|H - H_\infty| \to \infty$. Asymptotically, Eq. (11) reduces to;

$$6M(y) \sim \varphi_{pp}+5\varphi_p+6\varphi,$$

and so

$$\varphi \sim M(y). \tag{12}$$

Asymptotically, Eq. (10) then gives:

$$-\omega_\varphi \varphi_y \sim 2 - 2\frac{\varphi F_y}{\varphi_y}. \tag{13}$$

Now if $\varphi_y/\varphi F_y \to A = \text{const} \neq 0$ as $y \to \infty$ then $\ln \varphi \sim AF + \text{const}$, and we can clearly see that we cannot have both $F \to 0$ and $\varphi \to 0$ as $y \to 0$ as we have required for any finite A. Asymptotically, (9) gives:

$$-\omega_\varphi \varphi^2 - 12 \sim 6\omega\varphi_y. \tag{14}$$

If $\lim_{y\to 0} 1/(\omega\varphi_y) = 0$, then

$$-\frac{\omega_\varphi \varphi_y}{\omega} \sim 6,$$

and so $\omega \sim Ae^{-6y}$ for some A, and hence

$$\lim_{y\to\infty} \frac{1}{\omega\varphi_y} = \lim_{y\to\infty} \frac{e^{6y}}{\varphi}\frac{\varphi}{\varphi_y} \to \pm\infty$$

by $\lim_{y\to\infty} \varphi_y/\varphi = 0$. It follows then that we cannot have $\lim_{y\to\infty} 1/(\omega\varphi_y) = 0$ and hence, by Eq. (13), we cannot have $\lim_{y\to\infty} \varphi_y F_y/\varphi = 0$.

We must therefore have $\lim_{y\to\infty} \varphi/(\varphi_y F_y) = 0$, and so by both Eqs. (13) & (14):

$$\omega \sim -\frac{2}{\varphi_y}. \tag{15}$$

It follows from $\lim_{y\to\infty} \varphi_y/\varphi = \lim_{y\to\infty} M_y/M = 0$ that $\lim_{y\to\infty} |\varphi\omega| \to \infty$. In order to avoid ghosts we must have $\omega > -3/2$ and so, at late times, we must have $\varphi_y < 0$ so that $\omega \to +\infty$. It follows that $\varphi \to 0^+$. In this case the gravitational constant, $G = 1/\phi$, behaves as:

$$G(t) \sim G_\infty(1 + \varphi), \tag{16}$$

where $G_\infty = \phi_\infty^{-1}$. It follows that $G(t) > G_\infty$ at finite time and that the limiting value of G is approached from above.

3.2 Power-law approach: $\lim_{y\to\infty} M_y/M = -q$, $0 < q < \infty$

We now consider those cases where $M_y/M \to -q < 0$ as $y \to \infty$. This implies that $M(y) = f(y)e^{-qy}$ where $f_y/f \to 0$ as $y \to \infty$. If $K = \gamma = 0$ then this equates to $a^r |H - H_\infty| \to 0$ for all $r > q \neq \infty$ and $a^r |H - H_\infty| \to 0$ for all $r < q \neq 0$. In these cases, Eq. (11) gives:

$$\left((6-q)f(y) + f_y\right)e^{-qy} \sim \varphi_{yy} + 5\varphi_y + 6\varphi,$$

which, if $q \neq 6$, has the solution:

$$\varphi \sim c_0 e^{-2y} + c_1 e^{-3y} + \frac{(6-q)fe^{-qy}}{(3-q)(2-q)}.$$

If ($c_0 = c_1 = 0$), ($0 < q < 2$) and/or ($c_0 = 0$ and $2 < q < 3$) then from Eq. (10) we have

$$-qf(y)e^{-qy} - \left(q + q^2\right)\varphi = -q^2\omega\varphi^2. \tag{17}$$

It follows that:

$$\omega \sim \frac{12}{q(6-q)\varphi}. \tag{18}$$

If $c_0 \neq 0$ and $q > 2$ then:

$$\varphi \sim c_0 e^{-2y},$$

and from Eq. (10) we have:

$$\omega \sim \frac{3}{2\varphi}. \tag{19}$$

If $c_0 = 0$, $c_1 \neq 0$ and $q > 3$ then:

$$\varphi \sim c_1 e^{-3y},$$

and Eq. (10) gives:

$$\omega \sim \frac{4}{3\varphi}. \tag{20}$$

If $q = 2$ then:

$$\varphi \sim 4yM(y),$$

and Eq. (10) gives Eq. (19). If $q = 3$ then:

$$\varphi \sim -3yM(y),$$

and once again ω is described by Eq. (20).

If $q \neq 6$, we have found that the behaviour of ω is given either by Eq. (18), Eq. (19) or Eq. (20). In all three sub-cases $\varphi\omega \to$ const as $y \to \infty$, which is consistent with proposition 3.1. Additionally, we see that if we define $\lim_{r\to\infty} \varphi_y/\varphi = -r$, then if $0 < r < 6$ we have $\varphi \to 0^+$ and so as $t \to \infty$, $G(t)$ tends to its limiting value from above. If $r > 6$ then $\varphi \to 0^-$ and $G(t)$ tends to its limiting value from below.

We must still deal with the $q = 6$ case. If $q = 6$ then by Eq. (11):

$$f_y e^{-6y} \sim \varphi_{yy} + 5\varphi_y + 6\varphi,$$

where $f_y/f \to 0$. Now we therefore have $\varphi \sim c_0 e^{-2y} + c_1 e^{-3y} + g(y)e^{-6y}$ where g solves:

$$f_y \sim g_{yy} - 7g_y + 12g. \tag{21}$$

such that if $f_y = 0$ then $g = 0$. If $c_0 \neq 0$ or ($c_0 = 0$ and $c_1 \neq 0$) then we revert to one of the cases considered above and ω is given, respectively, by Eq. (19) or Eq. (20). We therefore take $c_0 = c_1 = 0$. It follows then from the requirement that $\lim_{\to\infty} f_y/f = 0$ that

$$\lim_{y\to\infty} (g_{yy}/f) = \lim_{y\to\infty} (g_y/f) = \lim_{y\to\infty} (g/f) = 0.$$

Thus, we have

$$\lim_{y\to\infty} \varphi_{yy}/M_y = \lim_{y\to\infty} \varphi_y/M_y = \lim_{y\to\infty} \varphi/M_y = 0.$$

Eq. (10) therefore gives:

$$\omega \sim \frac{6f(y)e^{-6y}}{\varphi_y^2} = \frac{6}{\varphi}\left[\frac{f(y)g(y)}{(g_y - 6g)^2}\right]. \tag{22}$$

So. if $g_y/g \to 0$, we have

$$|\varphi\omega| \to \frac{1}{6}\left|\frac{f}{g}\right| \to \infty,$$

and so ω diverges faster than $1/\varphi$. If $g_y/g \to -s$, with $s < 0$, we cannot have $f_y/f \to 0$ as required, and so $s > 0$. Thus,

$$|\varphi\omega| \to \frac{6}{(6+s)^2}\left|\frac{f}{g}\right| \to \infty,$$

and once again ω diverges faster than $1/\varphi$. Finally, if $g_y/g \to -\infty$, then we write $g = e^{-b(y)}$ where b, $b_y \to \infty$. Now $\lim_{y\to\infty}(yb_y)^{-1} = -\lim_{y\to\infty} b_{yy}/b_y^2 = 0$, and so $\lim_{y\to\infty} g_{yy}g/g_y^2 = 1$. By Eq. (21) we also have $g_{yy} \sim f_y$ and so:

$$|\varphi\omega| \sim \left|\frac{6fg}{g_y^2}\right| \sim \frac{6f}{g_{yy}} \sim \frac{6f}{f_y} \to \infty, \qquad \text{as}\, y \to \infty.$$

It follows that if $q = 6$ then ω diverges faster than $1/\varphi$.

We have shown that if $M_y/M \to -q$ then ω either diverges as $1/\varphi$ or, if $q = 6$, it is possible for ω to diverge faster than $1/\varphi$. In all cases then, we have, as proposed 3.1, that ω diverges faster than $|\varphi|^{-1+\epsilon}$ for all $\epsilon > 0$.

3.3 Fast approach: $M_y/M \to -\infty$

We now consider those cases where $M_y/M \to -\infty$ as $y \to \infty$. It follows from Eq. (11) that $\varphi \sim c_0e^{-2y} + c_1e^{-3y} + \bar{\varphi}$ where

$$M_y \sim \bar{\varphi}_{yy} \Rightarrow \bar{\varphi}_y \sim M. \tag{23}$$

If $c_0 \neq 0$, then Eq. (10) tells us that the behaviour of ω is described by Eq. (19). If $c_0 = 0$ and $c_1 \neq 0$ then the behaviour of ω is given by Eq. (20). In both cases ω diverges as φ^{-1}. We now consider the case $c_0 = c_1 = 0$ so $\varphi = \bar{\varphi}$. From Eq. (9), we have

$$-\omega_\varphi\varphi_y^2 \sim 12 + 2\omega\varphi_{yy}. \tag{24}$$

We define J by $J_y = M$ and $J \to 0$ as $y \to \infty$, and define $J = e^{-g(y)}$. It follows from $J_y/J \to -\infty$ and $J \to 0$ that g, $g_y \to \infty$. Now $\lim_{y\to\infty}(yg_y)^{-1} = -\lim_{y\to\infty} g_{yy}/g_y^2 = 0$ and so:

$$\frac{J_{yy}J}{J_y^2} = 1 + \frac{g_{yy}}{g_y}^2 \to 1. \tag{25}$$

Since $\varphi_y \sim M$ we have $\varphi \sim J$ and so, using the above relation, we have

$$(2\omega + \omega_\varphi\varphi) \sim -\frac{12\varphi}{\varphi_y^2}. \tag{26}$$

We now defined ϵ_0 by $|\varphi^{1-\epsilon}\omega| \to 0$ for all $\epsilon < \epsilon_0$ and $\to \infty$ for all $\epsilon > \epsilon_0$. We then have:

$$\omega \sim -\frac{12\varphi}{(1+\epsilon_0)\varphi_y^2}.$$

Since $\varphi_y/\varphi \to -\infty$, the above expression gives us $\lim_{y\to\infty}\varphi\omega = 0$. From the definition of ϵ_0 then, we must have $\epsilon_0 \geq 0$. Now clearly, for all $\epsilon > 0$, we have

$$\lim_{y\to\infty}\frac{\varphi^{\epsilon/2}}{y} = 0,$$

but then

$$\lim_{y\to\infty}\frac{\varphi^{\epsilon/2}}{y}=\frac{\epsilon}{2}\lim_{y\to\infty}\frac{\varphi_y}{\varphi^{1-\epsilon}}=0.$$

Thus, for all $\epsilon>0$ as $y\to\infty$, we have:

$$|\varphi|^{1-\epsilon}|\omega|=\left(\frac{\varphi_y}{\varphi^{1-\epsilon/2}}\right)^{-2}\to\infty.$$

Therefore, by the definition of ϵ_0 we must have $\epsilon_0\leq 0$, but since we also found that $\epsilon_0\geq 0$, and it then follows that $\epsilon_0=0$.

In this fast-approach case then, we have:

$$\omega\sim-\frac{12\varphi}{\varphi_y^2}. \tag{27}$$

Since $\varphi_y/\varphi\to-\infty$, we have that ω diverges faster than $1/\varphi$ in that limit, but since we also found that $\epsilon_0=0$, we have that $|\varphi^{1-\epsilon}\omega|\to\infty$ for all $\epsilon>0$, and so ω diverges faster than $|\varphi|^{-1+\epsilon}$ for all $\epsilon>0$.

We also note that for $\omega\to+\infty$, and hence to avoid ghosts, we must have $\varphi\to 0^-$ as $y\to\infty$. The effective gravitational 'constant' $G(t)\sim G_\infty(1+\varphi)$ therefore approaches its limiting value from below in the late-time limit.

We have shown that if $M_y/M\to-\infty$ then, as per proposition 3.1, ω diverges faster than $|\varphi|^{-1+\epsilon}$ for all $\epsilon>0$.

3.4 Summary

We have shown that proposition 3.1 holds in each of the three possible cases. We have therefore proved proposition 3.1. For there to be a de Sitter asymptote we therefore require that ω diverges faster than $|\phi_\infty-\phi|^{-1+\epsilon}$ for all $\epsilon>0$ as $y\to\infty$, $\phi\to\phi_\infty\neq 0$. Note that this is a more stringent condition than one might expect from considering linear scalar-tensor corrections to the metric around a spherically symmetric body. In the PPN formalism, scalar-tensor corrections to general relativity vanish as $\omega\to\infty$ and $\omega'/\omega^3\to 0$. The latter condition might lead one to expect a de Sitter cosmological limit if ω diverges faster than $1/\sqrt{|\phi_\infty-\phi|}$, however we have found that this is not the case. In summary: for a given theory of gravity, specified by $\omega(\phi)$, a de-Sitter limit for late-time Friedmann cosmology requires that there exists some $\phi_\infty\in(0,\infty)$ such that as $\phi\to\phi_\infty$

$$\omega\to\infty,\qquad \frac{\omega'}{\omega^{2+\epsilon}}\to 0,$$

for all $\epsilon>0$. If we define $-r=\lim_{y\to\infty}\varphi_y/\varphi$ then $G(t)\to G_\infty^+$ for $r<6$ and $G(t)\to G_\infty^-$ for $r>6$.

4 Decaying Gravity

In theories with asymptotically de Sitter behaviour, $G(t)\to\text{const}\neq 0$ in that limit. Another interesting limit to consider is that in which $G(t)\to 0$ at late times. In particular, when the equation of state is $\rho=-p$ an exact zero-curvature ($k=0$) Friedmann solution of Brans-Dicke

theory with this property is known. In this solution, $\omega = \text{const}$, $a \propto t^{\omega+1/2}$ and $G(t) \propto t^{-2}$. If $\omega \gtrsim 40,000$ today, as implied by tracking data for the Cassini spacecraft [19], then the possibility that the evolution of our universe is described by such a solution is not ruled out by observational or experimental evidence. Although the $G(t)$ evolution is strong, and appears to be ω independent, it is better to examine the $G(a) \propto a^{-4/(2\omega+1)}$ evolution, where the ω dependence appears explicitly and the $\omega \to \infty$ general relativity limit is evident. In order to classify the late-time behaviour of all such solutions, we define $G = G_0\chi(y)$, $H = H_0F(y)$, $k/a^2 = \gamma H_0^2 e^{-2y}$ where $8\pi G_0\rho_0/3H_0^2 = 1$; and as above, we have taken $y = \ln a$. With these definitions, the Einstein equations are now equivalent to:

$$\left(1 - \frac{\omega}{6}\frac{\chi_y^2}{\chi^2} - \frac{\chi_y}{\chi}\right) = \frac{\chi}{F^2}(1+K) - \frac{\gamma}{F^2}e^{-2y}, \tag{28}$$

$$\frac{2F_y}{F}\left(1 - \frac{\chi_y}{2\chi}\right) + 3 = \frac{3\chi(1+K_y/3+K)}{F^2} - \left(\frac{\omega}{2}+2\right)\frac{\chi_y^2}{\chi^2} + \frac{\chi_{yy}}{\chi} + 2\frac{\chi_y}{\chi} - \frac{\gamma e^{-2y}}{F^2}. \tag{29}$$

We can combine Eqs. (28) and (29) to give:

$$\frac{2F_y}{F}\left(1 - \frac{\chi_y}{2\chi}\right) + 6 = \frac{6\chi(1+K_y/6+K)}{F^2} - \frac{2\chi_y^2}{\chi^2} + \frac{\chi_{yy}}{\chi} + 5\frac{\chi_y}{\chi} - \frac{4\gamma e^{-2y}}{F^2}, \tag{30}$$

$$\frac{2F_y}{F}\left(1 - \frac{\chi_y}{2\chi}\right) = \frac{\chi K_y}{F^2} + \frac{2e^{-2y}}{F^2} + \frac{\chi_{yy}}{\chi} - \frac{\chi_y}{\chi} - (\omega+2)\frac{\chi_y^2}{\chi^2}. \tag{31}$$

We now divide the possible rates of change of $G(t)$ towards its asymptotic value into three classes and study each in turn.

4.1 Slow Approach: $G_y/G \to 0$

If $G \to 0$ more slowly than a^{-r} for any $r > 0$, then $\chi_y/\chi \to 0$ at late times, and Eq. (30) gives

$$\chi \sim F^2 + F_yF/3, \tag{32}$$

and so using $\lim_{y\to 0} \chi_y/\chi = 0$ we have:

$$\chi \sim F^2. \tag{33}$$

We can now use Eq. (31) to find $\omega(\phi)$:

$$\omega \sim -\frac{F}{F_y}\left(1 - \frac{K_yF}{4F_y}\right) \sim -\frac{2\chi}{\chi_y}\left(1 - \frac{K_y\chi}{2\chi_y}\right).$$

Now, assume that $\chi_y/\chi K_y \to \alpha$ for some α as $y \to \infty$. This would give $\chi \sim B\exp(\alpha K)$ for some $B \neq 0$. Given that $K \to 0^+$ as $y \to \infty$ and $\chi \to 0$, we cannot have $\alpha > -\infty$. Thus we must have $K_y\chi/\chi_y \to 0$ as $y \to \infty$, and so whatever K is:

$$\omega \sim -\frac{2\chi}{\chi_y}. \tag{34}$$

We note that as $y \to \infty$, $\omega \to \infty$, and also that

$$\omega' \sim \frac{G_0\omega^3}{4}\left(\chi_{yy} - \frac{\chi_y^2}{\chi}\right),$$

and so

$$\lim_{y\to\infty} \frac{\omega'}{\omega^3} = 0.$$

For these solutions:

$$G(t) \sim G_0 F^2 = \frac{G_0 H^2}{H_0^2} = \frac{3H^2}{8\pi\rho}.$$

Furthermore, it is clear that any scalar-tensor theory where $\omega \to \infty$ and $\omega'/\omega^3 \to 0$ as $\phi \to \infty$ will have such a late-time solution – since once $\omega(\phi)$ is known, the formula $\omega \sim -2\chi/\chi_y$ can be used to find χ_y and hence $H(y)$.

All bounds from local tests of gravity are satisfied in the limit $\omega \to \infty$ and $\omega'/\omega^3 \to 0$. Theories with stable late-time solutions of this class are therefore physically viable. We note that at late times,

$$\Omega_\Lambda = \frac{8\pi\rho G}{3H^2} \to 1,$$

however the spacetime is not not de Sitter since gravitation 'constant' is decaying i.e. $G \to 0$.

4.2 Power-law approach: $\lim_{y\to\infty} G_y/G = -q$

We now consider solutions where $a^r G \to 0$ for $r < q$ and $\to \infty$ for $r > q$. Eq. (30) gives:

$$(q+3)(q+2) \sim \frac{6\chi}{F^2} - \frac{4\gamma e^{-2y}}{F^2} - \frac{2F_y}{F}(1+q/2). \tag{35}$$

Since we cannot have $q < 0$, it follows that we must have $\lim_{y\to\infty} F_y/F = -r$ for some $r > 0$, and $\gamma e^{-2y}/\chi \to 0$ as $y \to 0$, which is certainly the case if $0 < q < 2$ and/or $\gamma = 0$, then we must have $F^2 \propto \chi$ in the $y \to \infty$ limit. Therefore, in this sub-case $2r = q$ and

$$\chi \sim \frac{(q+6)(q+2)F^2}{12}. \tag{36}$$

From Eq. (28), then we have $\lim_{y\to\infty} \omega = \omega_\infty$ and

$$\omega_\infty = \frac{2}{q} - \frac{1}{2} > -1/2. \tag{37}$$

If $|\gamma e^{-2y}/\chi| \to \infty$, which certainly requires $q \geq 2$ and $\gamma \neq 0$, Eq. (30) gives:

$$(q+3-r)(q+2) \sim \frac{-4\gamma e^{-2y}}{F^2}.$$

Since $q > 2$, we must therefore have $r = 1$ and also $F^2 \sim -4\gamma e^{-2y}/(q+2)^2$. For such solutions to exist we must therefore have $\gamma < 0$. From Eq. (28) we have:

$$1 + q - \frac{\omega q^2}{6} \sim \frac{(q+2)^2}{4},$$

Thus, we have $\lim_{y\to\infty} \omega = \omega_\infty$ where

$$\omega_\infty = -\frac{3}{2}.$$

Finally, if $q=2$ and $\chi e^{2y}\to a_0=\mathrm{const}\neq 0$, then Eq. (30) gives $r=1$ and

$$\frac{3}{2}\chi\sim\gamma e^{-2y}+4F^2. \tag{38}$$

Hence, we must have $3a_0/2>\gamma$. We define $\gamma=b_0a_0$, and then from Eq. (28) we have $\lim_{y\to\infty}\omega=\omega_\infty$, where

$$\omega_\infty=\frac{3(1+2b_0)}{2(3-2b_0)}>-\frac{3}{2}. \tag{39}$$

Observationally acceptable solutions may require that $\omega>40,000$ today. For $\omega_\infty>40000$ we would need $q<5\times 10^{-5}$.

4.3 Fast approach: $|G_y/G|\to\infty$

The final class of solutions that we consider have $G\to 0$ more quickly than a^{-r}, for any $r>0$. This implies that $|\chi_y/\chi|\to\infty$. For such solutions we write $\chi=e^{-g(y)}$ and we require that g, $g_y\to\infty$. This further implies that $\chi_{yy}\sim\chi_y^2/\chi$. We also write $F=\exp(-f(y))$. If $\gamma=0$ then Eq. (30) gives

$$(f_y-g_y)\,g_y\sim -6\exp(2f-g). \tag{40}$$

We define J, by $J_y=F$ and $J\to 0$ as $y\to\infty$. We note that $J\sim F^2/F_y$ as $y\to\infty$ since $F_{yy}F/F_y^2\to 1$. We now make the ansatz:

$$e^g\sim\lambda J^2.$$

With this definition of λ:

$$g_y^2\sim 4\lambda e^{2f-g}$$

Since $g_y=2J_y/J$ and $F_y=J_{yy}$, we have

$$\frac{f_y}{g_y}=\frac{J_{yy}J}{2J_y^2}\sim\frac{1}{2}.$$

Eq. (40) then gives

$$g_y^2\sim 12e^{2f-g}. \tag{41}$$

It follows that $\lambda=\sqrt{3}$ is required. In this case then:

$$\chi\sim\frac{\sqrt{3}F^2}{F_y}.$$

Using Eq. (28) we find:

$$(1+\frac{2\omega}{3})\to 0$$

and so as $y\to\infty$,

$$\omega\to\omega_\infty=-\frac{3}{2}. \tag{42}$$

If $\gamma\neq 0$, Eq. (30) gives $f\sim y+h$ where:

$$(h_y-g_y)\,g_y\sim 4\gamma\exp(2h) \tag{43}$$

We define $L_y = e^h$ and so as $y \to \infty$, $L \to \infty$. Solutions to this equation require $g \sim \mu L$. It follows from this that $h_y/g_y \to 0$, and so $\mu = \sqrt{-4\gamma}$. Since we must have $g \to \infty$, such solutions only exist is $\gamma < 0$. In this case then:

$$\chi = e^{-g} \sim e^{-\mu L}. \tag{44}$$

From Eq. (28), we find:

$$-\frac{\mu^2 \omega}{6} \sim -\gamma = \mu^2/4,$$

and so as $y \to \infty$: $\omega \to \omega_\infty = -3/2$.

4.4 Suppressing "Boltzmann Brains"

We have shown that solutions in which the gravitational constant decays at late times i.e. $G \to 0$, require that, as $G^{-1} \sim \phi \to \infty$ either $\omega \to \infty$ or $\omega \to \omega_\infty = \text{const}$. Generally speaking, the smaller $\lim_{\phi\to\infty} \omega$ is, the faster G decays at late times. The fastest decay occurs when $\omega_\infty = -3/2$. Observational data may require that$\omega > 40,000$, today [19], and so solutions with small values of ω_∞ are unlikely to exist in physically viable theories. Theories with very large values of ω_∞ are, however, permitted alternatives to standard general relativity.

The decay of G in similar scalar-tensor theories would occur over time scales that are large compared to the Hubble time, and at late finite times the spacetimes they predict would be observationally similar to de Sitter, although both G and H would be decaying with time. This is an interesting scenario. If space-time is asymptotically de Sitter, then $H \to H_0 \neq 0$ at late times. Since the expansion is accelerating, there exists an event horizon, and as is the case with a black hole horizon, the de Sitter horizon has a temperature: $T = H_0/2\pi = \text{const} \neq 0$. The presence of a non-zero minimum temperature in space-times that asymptote to de Sitter is important, as it means that even very rare thermal fluctuations are eventually expected to occur. For instance, the probability that a certain thermal fluctuation occurs goes as $e^{-S/T}$ for some action S which does not depend on T. The 4-volume of space-time grows as $\int a^3 \mathrm{d}t$, which in de Sitter spacetime grows as $e^{3H_0 t}$. Thus, after a certain time t, the number of times that a particular thermal fluctuation is expected to have occurred is $n \sim e^{-S/T+3Ht}$. Provided $T \to \text{const}$, n will eventually be greater than unity, and $n \to \infty$ asymptotically. A particularly extreme and topical example of this is the spontaneous emergence of "Boltzmann brains" in general -relativistic space-times that asymptote to de Sitter. Self-aware "observers" could emerge from the vacuum as a result of thermal fluctuations [20]. In an eternal, asymptotically de Sitter, universe such "observers" should vastly outnumber "ordinary" carbon-based observers such as ourselves. This conclusion is changed in scalar-tensor cosmologies.

In scalar-tensor gravity, we have seen that theories in which either $\omega \to \omega_\infty > 40,000$, or $\omega \to \infty$ as $\phi \to \infty$, admit asymptotic solutions that would be observationally indistinguishable from de Sitter today, but in which $G, H \to 0$ at late times. Since G and H would, in such theories, decay only very slowly over a Hubble time, we still have $T \approx H/2\pi$. The 3-volume of a spatial slice of constant t goes like $a^3/H^3 \propto e^{3y-3\ln F}$, and so the number of thermal fluctuations that are expected goes like:

$$n \sim \int^t e^{-S/T(t)+3y-3\ln F} \mathrm{d}t \propto \int^y \left(e^{-\frac{2\pi S}{H_0 F(y)}+3y-4\ln F} \right) \mathrm{d}y \propto \int^y \left(e^{-\frac{2\pi S}{F(y)}+3y} \right) \mathrm{d}y. \tag{45}$$

where, as above, $F = H/H_0$ and $y = \ln a$.

We have found that in all theories where $\omega \to \infty$ or $\omega \to \omega_\infty > 40,000$ as $\phi \to \infty$ at late times, $F \to 0$ slower than e^{-qy}, where $q \lesssim 5\times 10^{-5}$. Even with this strong restriction, however, there is still a class of theories for which $F \to 0$ faster than $1/y$ as $y \to \infty$. Specifically, from Eqs. (33)-(34) we find that this will be the case if, as $\phi \to \infty$, $\omega(\phi)$ grows more slowly than $\phi^{1/2}$; Brans-Dicke theories with $\omega = \text{const}$ certainly satisfy this constraint. In these theories, as $y \to \infty$, the integrand in Eq. (45) would asymptotically tend to zero faster than e^{-ry} for any r. Thus, $n \to n_\infty = \text{const} < \infty$ at late times. If S is small enough, then $n_\infty \ll 1$. In these cases, even though the universe would still be eternal, very rare thermal fluctuations like "Boltzmann brains" would not be expected to occur even once.

This is another example (see also [21]), of how, if some or all of the traditional constants of Nature vary slowly with time, we cannot use current observations of the universe to make definitive statements about the expected behaviour of the universe in the far future.

5 Conclusions

We have investigated two features of the general behaviour of scalar-tensor gravity theories. Motivated by the need to understand the possible origins of de Sitter expansion in the early and late periods of the universe's history we have investigated how it can arise in general scalar-tensor gravity theories. We considered Friedmann universes filled with a mixture of a vacuum stress with equation of state $p = -\rho = -\rho_0$ and other fluids which have total energy density $\rho_1 = \rho_0 K(y)$ and pressure p_1, where $y = \ln a$ and $\lim_{y\to\infty} K(y) = \lim_{y\to\infty} K_y(y) = 0$,so the vacuum stress dominates at late times. In scalar-tensor theories with a coupling function $\omega(\phi)$, we find that there is asymptotic approach to de Sitter expansion at late times, where $\phi \to \phi_\infty$, provided the coupling function $\omega(\phi)$ diverges faster than $|\phi_\infty - \phi|^{-1+\epsilon}$ for all $\epsilon > 0$ as $\phi \to \phi_\infty \neq 0$. This means that, for a given theory, specified by $\omega(\phi)$, there must exist some $\phi_\infty \in (0, \infty)$ such that $\omega \to \infty$ and $\omega'/\omega^{2+\epsilon} \to 0$ as $\phi \to \phi_\infty$ in order for cosmological solutions of the theory to approach de Sitter expansion at late times. This differs from the conditions required to establish a general relativity limit that has vanishing corrections to the weak-field PPN corrections to general relativity in the solar system: $\omega \to \infty$ and $\omega'/\omega^3 \to 0$.

Brans-Dicke theory ($\omega = \text{const}$) does not emit a de Sitter limit in the presence of $p = -\rho$ stress. There is instead power-law inflation and $G \propto t^{-2}$. With this behaviour in mind we analysed the possible late time evolution of $G(t)$ in the Friedmann cosmological models of scalar-tensor theories defined by an arbitrary $\omega(\phi)$ and divided them into three classes depending upon the rate of decay of G with the expansion scale factor. The scenarios in which G decays over quickly (i.e. over a Hubble time or faster) would be difficult to realize in a manner that was compatible with solar system tests of gravity i.e. $\omega > 40,000$ today. The subset of theories with a physically viable slow decay were found to be particularly interesting because if, as is the case in Brans-Dicke theory, $\omega \sim o(\phi^{1/2})$ as $\phi \to \infty$, then the expected number of extremely rare thermal fluctuations, $n(t)$, that occur within the visible universe, after a time t, would asymptote to a constant value. In general relativistic de Sitter space-time, $n(t) \to \infty$, which has led some to postulate that a typical 'observer' of our Universe would most likely have arisen out of the vacuum as a thermal fluctuation. The intrinsic probability of such a "Boltzmann brain" fluctuation is tiny, but since $n(t) \to \infty$ in an eternal de Sitter universe, the number of such observers would grow without bound. In scalar-tensor theories, like Brans-Dicke, we have shown that the slow decay of $G(t)$ can prevent this strange situation from occurring with any significant probability.

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References

[1] G. Lemaître, Proc. Nat. Acad. Sci. 20, 12 (1933).

[2] F. Hoyle, Mon. Not. R. astr. Soc.,108, 372 (1948).

[3] W.H. McCrea, Proc. R. Soc. 206, 562 (1951).

[4] F. Hoyle and J. Narlikar, Proc. Roy. Soc. A273, 1 (1963).

[5] J.D. Barrow, Phys. Lett. B., 180, 335 (1987) and J.D. Barrow, Nucl. Phys. B., 310, 743 (1988).

[6] A. Starobinsky, Phys. Lett. 91B, 99 (1980).

[7] B. Whitt, Phys. Lett. B. 215 176 (1984); J.D. Barrow and S. Cotsakis, Phys. Lett. B. 214, 515 (1988); K. Maeda, Phys. Rev. D 39, 3159 (1989).

[8] J.D. Barrow and S. Hervik, Phys. Rev. D 73, 023007 (2006) and J.D. Barrow and S. Hervik, Phys. Rev. D74, 124017 (2006).

[9] C. Brans and R. Dicke, Phys. Rev. 124, 925 (1961).

[10] C. Mathiazhagen and V.B. Johri, Class. Quantum Grav. 1, L29 (1984).

[11] D. La and P. Steinhardt, Phys. Rev. Lett. 62, 376 (1989).

[12] J.D. Barrow and K-I. Maeda, Nucl. Phys. B. 341, 294 (1990).

[13] J.D. Barrow, Mon. Not. R. astr. Soc., 282, 1397 (1996).

[14] K. Nordvedt, Ap. J. 161, 1059 (1970).

[15] J.D. Barrow and J. Mimoso, Phys. Rev. D 50, 3746 (1994)

[16] G.L. Comer, N. Deruelle and D. Langlois, Phys. Rev. D 55, 3497 (1997).

[17] J.D. Barrow and P. Parsons, Phys. Rev. D 55, 1906 (1997).

[18] J.D. Barrow, Phys. Rev. D 47, 5329 (1992).

[19] B. Bertotti, L. Iess, and P. Tortora, Nature 425, 374 (2003). C.M. Will, Living Rev. Rel. 4, 4 (2001).

[20] L. Boltzmann, Nature 51, 413 91895); J.D. Barrow and F.J. Tipler, *The Anthropic Cosmological Principle*, Oxford UP, Oxford (1986), chap. 3.8; M. J. Rees, Before the Beginning: Our Universe and Others (Helix Books, 1997), p. 221; L. Dyson, M. Kleban, and L. Susskind, JHEP 0210, 011 (2002); A. Albrecht and L. Sorbo, Phys. Rev. D70 (2004). D. Page, hep-th/0612137.

[21] S. Carlip, arXiv: hep-th/0703115v5